Exercise 16

Convert each of the following Fredholm integral equation in 9–16 to an equivalent BVP:

$$u(x) = e^x + 1 + \int_0^1 K(x,t)u(t)\,dt,\ K(x,t) = \begin{cases} 2t, & \text{for } 0 \le t \le x \\ 2x, & \text{for } x \le t \le 1 \end{cases}$$

Solution

Substitute the given kernel $K(x,t)$ into the integral.

$$u(x) = e^x + 1 + \int_0^x 2tu(t)\,dt + \int_x^1 2xu(t)\,dt \tag{1}$$

Differentiate both sides with respect to x.

$$u'(x) = e^x + \frac{d}{dx}\int_0^x 2tu(t)\,dt + \frac{d}{dx}\int_x^1 2xu(t)\,dt$$

Apply the Leibniz rule to differentiate the second integral.

$$\begin{aligned} &= e^x + 2xu(x) + \int_x^1 \frac{\partial}{\partial x}2xu(t)\,dt + 2xu(1)\cdot 0 - 2xu(x)\cdot 1 \\ &= e^x + \int_x^1 2u(t)\,dt \\ &= e^x - 2\int_1^x u(t)\,dt \end{aligned} \tag{2}$$

Differentiate both sides with respect to x once more.

$$\begin{aligned} u''(x) &= e^x - 2\frac{d}{dx}\int_1^x u(t)\,dt \\ &= e^x - 2u(x) \end{aligned}$$

The boundary conditions are found by setting $x = 0$ and $x = 1$ in equations (1) and (2), respectively.

$$\begin{aligned} u(0) &= e^0 + 1 + \int_0^0 2tu(t)\,dt + \int_0^1 2(0)u(t)\,dt = 2 \\ u'(1) &= e^1 - 2\int_1^1 u(t)\,dt = e \end{aligned}$$

Therefore, the equivalent BVP is

$$u'' + 2u = e^x,\ u(0) = 2,\ u'(1) = e.$$